TOPIC 4: Differentiation

A. DERIVATIVES

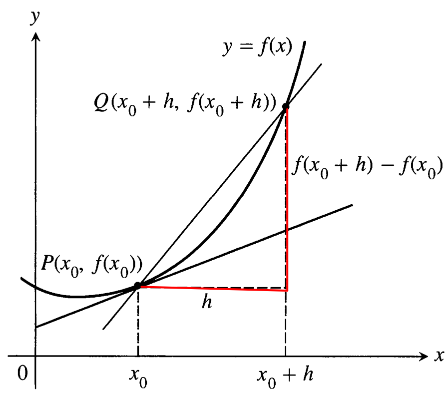
1. DEFINITION OF DERIVATIVE

<u>Definition:</u>

If $f(x)$ is a function defined on an open interval $I=(a,b)$ that contains the point c, the derivative of f at c, denoted by $f'(c)$, is

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad ....(*)$$

provided the limit exists.



The slope of the tangent line at *P* is

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

If f is differentiable at all $x\in I$, we say that f is differentiable on I. In this case, we can define a new function f' on I where $f'(x)$ assumes the value of the derivative of f at $x\in I$. We call f' the derivative of f, and we also write $f'(x)=\dfrac{df}{dx}$ or $\dfrac{d}{dx}(f)$.

<u>Other notations</u>

Letting $x=c+h$ in equation (*), which is $f'(c)=\lim_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$, then equation (*) becomes $f'(c)=\lim_{h\to 0}\dfrac{f(x)-f(c)}{x-c}$.

Equation (*) rewritten with *x* instead of *c*, and Δx instead of *h*, becomes

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The numerator $f(x+\Delta x)-f(x)$ represents the change in the value of *f* when one changes the argument *x* by a small amount Δx.

We let $\Delta y=f(x+\Delta x)-f(x)$ denote the change in the y-value. Then $f'(x)=\lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$

Leibniz's notation: $\dfrac{dy}{dx}=\lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$

Different notations are used to represent the derivative of a function $y = f(x)$ with $f'(x)$ being the most common. Some others are $\frac{df}{dx}, y', \frac{dy}{dx}, \frac{d}{dx}f(x), Df(x)$, and $D_x f(x)$.

One interpretation of the derivative of a function at a point is the **slope of the tangent line** at this point. The slope of the tangent line at the point $(c, f(c))$ on the graph of $y = f(x)$ is $f'(c)$, the derivative of f at c. Another interpretation: $f'(c)$ is also the **instantaneous rate of change** of y with respect to x at $x = c$; sometimes $\left.\frac{dy}{dx}\right|_{x=c}$ or $\left.\frac{dy}{dx}\right]_{x=c}$ is used to denote $f'(c)$.

The notation $\frac{dy}{dx}$ is referred to as the **derivative of *y* with respect to *x***.

Example:

Given the graph of the function $f(x) = 3x^2$ (which is a parabola), use the definition of the derivative to obtain the slope of the tangent to the graph at the point (2, 12).
[How do yu know that (2, 12) lies on the graph?]

Solution:
To find $f'(2)$.

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$\frac{f(2+h)-f(2)}{h} = \frac{3(2+h)^2-12}{h} = \frac{3(4+4h+h^2)-12}{h} = \frac{12h+3h^2}{h} = 12+3h$$

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}(12+3h) = 12$$

[Can you obtain an equation for the tangent to the graph at the point (2, 12)?]

[***Reminder***: Try doing it for another point on the graph, say, one with $x = -1$.
Then try doing it for a general point $(a, \_?\_\_)$.]

Example:
Use the definition for derivative to find the derivative of $y = \sqrt{x}$ for $x > 0$.
Solution: Let $f(x) = \sqrt{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x) = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

In short, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Example:
Is the function $f(x) = |x|$ differentiable at $x = 0$? Let's see.

[We cannot take for granted that every function we see is differentiable everywhere.]

Continuity and Differentiability
Theorem: If a function f is differentiable at a number c, then it is continuous at c.
In other words, if c is a discontinuity of f, then f is not differentiable at c.

Differentiability implies continuity.

2. DERIVATIVE RULES

1	Derivative of a Constant Function	$\frac{d}{dx}k = 0$	$\frac{d}{dx}k = 0$	
2	Power Rule	$\frac{d}{dx}x^n = nx^{n-1}, n \in R$	$\frac{d}{dx}x^n = nx^{n-1}, n \in R$	
3	Constant Multiple Rule	$\frac{d}{dx}(kf) = k\frac{df}{dx}$, where k is a constant.	$\frac{d}{dx}(ku) = k\frac{du}{dx}$	$(ku)' = ku'$
4	Sum Rule	$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$	$(u+v)' = u' + v'$
5	Product Rule	$\frac{d}{dx}(f \cdot g) = f\frac{dg}{dx} + g\frac{df}{dx}$	$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$	$(u \cdot v)' = u'v + uv'$
6	Quotient Rule	$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$ for x with $g(x) \neq 0$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

[** If you have studied calculus before, you are probably more familiar with only one of the three columns on the right.]

[** Some authors or teachers like to include the power rule and the root rule; these are not really necessary after you master the chain rule.]

Example:
Find the derivatives of the following functions.

a) $y = x^4 + 12x - 4$ b) $f(x) = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$ c) $g(t) = \frac{t^2 - 1}{t^2 + 1}$

d) $y = \frac{(x-1)(x^2 - 2x)}{x^4}$ e) $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

Solution:

a) $y = x^4 + 12x - 4$

$$\frac{dy}{dx} = \frac{d}{dx}x^4 + \frac{d}{dx}12x + \frac{d}{dx}(-4)$$
$$= 4x^3 + 12$$

b) $f(x) = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$

$$f'(x) = \frac{1}{x}\frac{d}{dx}\left(x^2 + \frac{1}{x}\right) + \left(x^2 + \frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= x^{-1}\left(\frac{d}{dx}x^2 + \frac{d}{dx}x^{-1}\right) + \left(x^2 + x^{-1}\right)\frac{d}{dx}x^{-1}$$
$$= x^{-1}\left(2x - x^{-2}\right) + \left(x^2 + x^{-1}\right)\left(-x^{-2}\right)$$
$$= 2 - x^{-3} - 1 - x^{-3}$$
$$= 1 - 2x^{-3}$$

[*The steps shown here are to illustrate the derivative rules; it would be easier if you rewrite* $f(x)$ *as* $x + \frac{1}{x^2}$.]

d) $y = \frac{(x-1)(x^2-2x)}{x^4}$

$$\frac{dy}{dx} = \frac{x^4\frac{d}{dx}\left[(x-1)(x^2-2x)\right] - (x-1)(x^2-2x)\frac{d}{dx}x^4}{\left(x^4\right)^2}$$
$$= \frac{x^4\left[(x-1)\frac{d}{dx}(x^2-2x) + (x^2-2x)\frac{d}{dx}(x-1)\right] - (x-1)(x^2-2x)\frac{d}{dx}x^4}{x^8}$$
$$= \frac{x^4\left[(x-1)(2x-2) + (x^2-2x)(1)\right] - (x-1)x(x-2)\left(4x^3\right)}{x^8}$$
$$= \frac{(x-1)(2x-2) + (x^2-2x) - (x-1)(x-2)\left(4\right)}{x^4}$$
$$= \frac{-x^2 + 6x - 6}{x^4}$$

[Here, we apply he quotient rule and also the product rule. *Is there a simpler way*?]

Derivatives of Trigonometric Functions and Their Inverses

For any x for which a trigonometric functions or its inverse is defined, we have the following formulae:

(Note that these formulae are valid only when x is measured in radians.)

$\frac{d}{dx}\sin x = \cos x.$	(*) $\frac{d}{dx}\cot x = -\csc^2 x.$
$\frac{d}{dx}\cos x = -\sin x.$	(*) $\frac{d}{dx}\sec x = \sec x \tan x.$
$\frac{d}{dx}\tan x = \sec^2 x.$	(*) $\frac{d}{dx}\csc x = -\csc x \cot x.$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert < 1.$	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}.$
$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}, \quad \lvert x\rvert < 1.$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}, \quad \lvert x\rvert > 1.$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$	$\frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}, \quad \lvert x\rvert > 1.$

[** You are required to remember the three forms shaded; those marked with (*) can be obtained by using differentiation rules. The others will be provided when required.]

[**Derivatives of sin and cos are obtained from definition using limits; $\frac{d}{dx}\tan x = \sec^2 x$. can be obtained using the quotient rule. Most people would take the trouble to memorize this because it crops up quite often in problems.]

Example:

Find the derivatives of the following functions.

a) $y = \tan x$ b) $y = x^2 \sin x + 2x\cos x - 2\sin x$ c) $y = \frac{\cos x}{1+\sin x}$

d) $y = \frac{\sec x}{1+\tan x}$ e) $y = \sin x + 10\tan x$ f) $y = x^2 \cot x - \frac{1}{x^2}$

Solution:

a) $y = \tan x$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

<u>Derivatives of Logarithmic and Exponential Functions.</u>

1. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ (for $x > 0$); $\frac{d}{dx}\ln|x| = \frac{1}{x}$ (for $x \neq 0$)
2. $\frac{d}{dx}e^x = e^x$

[** Some authors/teachers like to include

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a \text{, where } a \text{ is a positive constant}$$

as rule for students to memorize.

This is *not really necessary if you already knew* $a^x = e^{x\ln a}$]

3. THE CHAIN RULE

If g is differentiable at the point c and f is differentiable at the point $g(c)$, then the composite function $f \circ g$ is differentiable at c, and

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c)$$

In **Leibniz's** notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Again, some authors/teachers like to include the following two formulas for students to memorize.

1. $\frac{d}{dx}\ln g(x) = \frac{1}{g(x)} \cdot \frac{d}{dx}g(x)$ and 2. $\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot \frac{d}{dx}g(x)$

[** <u>*These two items are just special cases of the chain rule*; there is no real need to memorize remember</u>. Practise using the chain rule to derive these two formulas.]

For #1, we let $f(u) = \ln u$ and $u = g(x)$; for #2, we let $f(u) = e^u$ and $u = g(x)$.

<u>Example</u>: How to determine $\frac{d}{dx}\left(x^2+1\right)^5$? [*Could this be done without the chain rule*?]

$\left(x^2+1\right)^5 = (f \circ g)(x) = f(g(x))$ where $f(x) = x^5$ and $g(x) = x^2+1$.

$(f \circ g)(x) = f\left(g(x)\right) = f(x^2+1) = \left(x^2+1\right)^5$

Let $y = \left(x^2+1\right)^5$, i.e., $y = (f \circ g)(x)$

What is $\frac{dy}{dx}$ or $(f \circ g)'(x)$?

$f(x) = x^5$ and $g(x) = x^2+1$. $f'(x) = 5x^4$ and $g'(x) = 2x$. $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ $= f'(x^2+1) \cdot 2x$ $= 5(x^2+1)^4 \cdot 2x$ $= 10x(x^2+1)^4$	Let $u = x^2+1$ and $y = u^5$. [$u = g(x)$, $y = f(u)$] Then $y = u^5 = (x^2+1)^5$ $\frac{dy}{du} = 5u^4$, $\frac{du}{dx} = 2x$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot 2x = 5(x^2+1)^4 \cdot 2x = 10x(x^2+1)^4$

Example:
Find the derivatives of the following functions.

a) $y=(2x+1)^5$ b) $y=\left(\frac{x^2}{8}+x-\frac{1}{4}\right)^4$ c) $y=\sin(x^2+x)$

d) $y=\ln(x^3+1)$ e) $y=e^{2x-x^2}$ f) $y=\left(e^{2x}+e^{x}\right)\ln 2x$

g) $y=10^{x^2}$ h) $y=\sin(\cos(\tan x))$ i) $y=\frac{x}{\sqrt{x^2-2x+3}}$ j) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$

Solution:

a) $y=(2x+1)^5$

Let $u=2x+1$ Then $y=u^5$.

$\frac{du}{dx}=2$ and $\frac{dy}{du}=5u^4$.

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$=5u^4(2)$$
$$=10(2x+1)^4$$

Here 's a very important training

(a) Obtain the derivatives of the following functions of x by using the chain rule, with all the steps properly written.

$$\sin 3x,\ \cos 3x,\ \tan 3x,\ e^{3x},\ \ln(3x)$$

(b) Obtain the derivatives of the following functions of x by using the chain rule, without writing down detailed steps.

$$\sin 3x,\ \cos 3x,\ \tan 3x,\ e^{3x},\ \ln(3x)$$

4. HIGHER DERIVATIVES

For a differentiable function f, the derivative f' is also a function.

If f' is also differentiable, its derivative is denoted by

$$(f')' = f''..$$

It is called the **second derivative** of f.

Using Leibniz's notation, the second derivative of $y=f(x)$ is $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$

There are several ways of writing the second derivative.

$$f''(x)=\frac{d}{dx}(f'(x))=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\frac{dy'}{dx}=y''$$

In general, the n-th derivative of f with respect to x for any positive integer *n*, denoted by $f^{(n)}(x)=\frac{d^n}{dx^n}f$, is defined as $f^{(n)}(x)=\frac{d}{dx}\frac{d^{n-1}}{dx^{n-1}}f$, i.e,. as the (first) derivative of the $(n-1)$ th derivative of f with respect to x.

Example:

Find y'' and y''' for

a) $y = x^3 - 7x^2 + 100x + 1$ b) $y = x\cos x$

Solution:

a) $y = x^3 - 7x^2 + 100x + 1$

First Derivative: $y' = 3x^2 - 14x + 100$

Second derivative: $y'' = 6x - 14$

Third derivative: $y''' = 6$

5. IMPLICIT DIFFERENTIATION

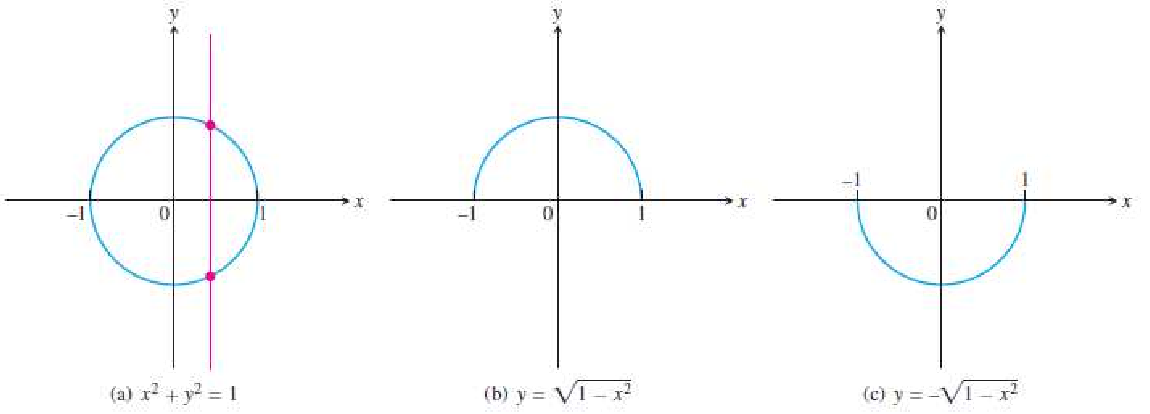
The function that we have looked at in the earlier sections can be described by expressing one variable (the ***dependent*** variable) ***explicitly*** in terms of another variable (the ***independent*** variable) .

An explicit function is given in the form $y = f(x)$, where *y* is the ***dependent*** variable and *x* is the ***independent*** variable.

Examples: $y = \sqrt{x^2 + 3}$, $y = \cos x$, $y = \begin{cases} x^2 & \text{if } x < 2 \\ 4x - 4 & \text{if } x \geq 2 \end{cases}$

Some functions are defined ***implicitly*** by a relation between *x* and *y* through an equation involving *x* and *y*.

Example:



(a) The circle is not the graph of a function; it fails the vertical line test.

(b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$

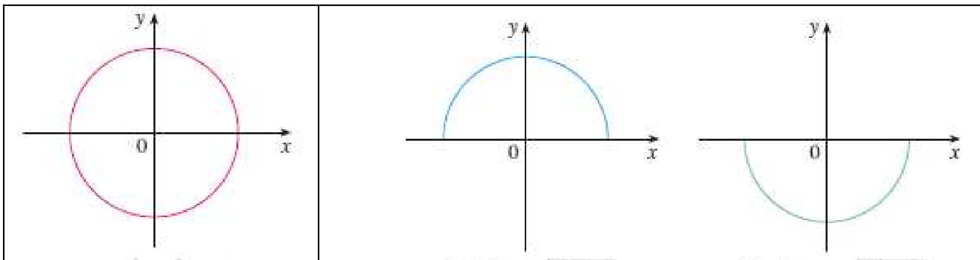
(c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$ (T*)

Given the equation $x^2 + y^2 = 1$ and solving for *y* in terms of *x*, we get $y = \pm\sqrt{1 - x^2}$.

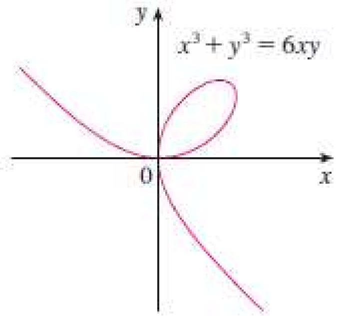
The equation $x^2 + y^2 = 1$ implicitly defines two functions of x,

i.e., $f(x) = \sqrt{1 - x^2}$, for $-1 \leq x \leq 1$ and $g(x) = -\sqrt{1 - x^2}$, for $-1 \leq x \leq 1$

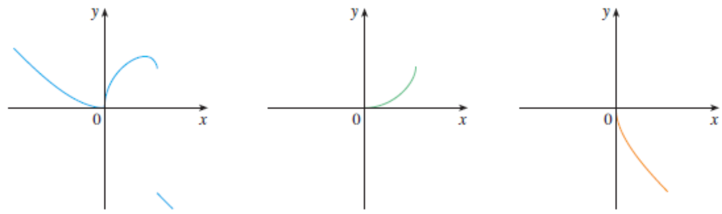
Examples:



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$ (S*)



The folium of Descartes



Graphs of three functions defined by the folium of Descartes (S*)

An equation in *x* and *y* in the form $F(x, y) = 0$ (or equivalent) may implicitly define one or more functions of *x*; but it is not always possible or easy to put in the form $y = f(x)$ and then to further obtain $\frac{dy}{dx}$ in the usual way.

We apply ***implicit differentiation*** when a variable is defined implicitly as a function of another variable.

To find the derivative of *y* with respect to *x*, we do not need to solve for *y* as a function of *x*. Instead, we use implicit differentiation. We treat *y* as a differentiable function of *x* and differentiate both sides of the equation with respect to *x*; then solve for $\frac{dy}{dx}$ in terms of *x* and *y*.

Example:

(a) Given $x^2 + y^2 = 1$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle at the point $\left(\frac{3}{5}, \frac{4}{5}\right)$.

Solution

(a) Differentiate both sides of the equation $x^2 + y^2 = 1$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

[Using the chain rule, $\frac{d}{dx}(y^2)=\frac{d}{dy}(y^2)\cdot\frac{dy}{dx}=2y\frac{dy}{dx}$]

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

(b) At the point $\left(\frac{3}{5},\frac{4}{5}\right)$, we have $x=\frac{3}{5}, y=\frac{4}{5}$.

So, $\frac{dy}{dx}=-\frac{\frac{3}{5}}{\frac{4}{5}}=-\frac{3}{4}$; this is the slope of the tangent at the point $\left(\frac{3}{5},\frac{4}{5}\right)$.

An equation of the tangent to the circle at the point $\left(\frac{3}{5},\frac{4}{5}\right)$ is given by

$$y-\frac{4}{5}=-\frac{3}{4}\left(x-\frac{3}{5}\right) \quad \text{or} \quad 3x+4y-5=0$$

Example:

Find $\frac{dy}{dx}$ for

a) $x^2y-xy^2+x^2+y^2=0$ b) $y^2=x^2+\sin xy$ c) $y=x^x$.

Solution:

a) $x^2y-xy^2+x^2+y^2=0$

$$\frac{d}{dx}(x^2y)-\frac{d}{dx}(xy^2)+\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(0)$$

$$x^2\frac{d}{dx}(y)+y\frac{d}{dx}x^2-x\frac{d}{dx}(y^2)-y^2\frac{d}{dx}(x)+2x+\frac{d}{dx}(y^2)=0$$

$$x^2\frac{dy}{dx}+y(2x)-x\left(2y\frac{dy}{dx}\right)-y^2(1)+2x+2y\frac{dy}{dx}=0$$

$$\left(x^2-2xy+2y\right)\frac{dy}{dx}=y^2-2xy-2x \qquad \therefore \frac{dy}{dx}=\frac{y^2-2xy-2x}{x^2-2xy+2y}$$

b) $y^2=x^2+\sin xy$

$$\frac{d}{dx}(y^2)=\frac{d}{dx}(x^2)+\frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx}=2x+\cos xy\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx}=2x+(\cos xy)(y+x\frac{dy}{dx})$$

$$2y\frac{dy}{dx}-(\cos xy)(x\frac{dy}{dx})=2x+(\cos xy)y$$

$$(2y-x\cos xy)\frac{dy}{dx}=2x+y\cos xy$$

$$\therefore \frac{dy}{dx}=\frac{2x+y\cos xy}{2y-x\cos xy}$$

c) $y = x^x$

Taking logarithms of both sides, $\ln y = \ln x^x$

$$\ln y = x \ln x$$

$$\frac{d}{dx}\ln y = \left(\frac{d}{dx}x\right)\ln x + x\left(\frac{d}{dx}\ln x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 \qquad \therefore \frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

6. MAXIMUM AND MINIMUM VALUES

Definition: Absolute Extreme Values

Let f be a function with domain D and $c \in D$.

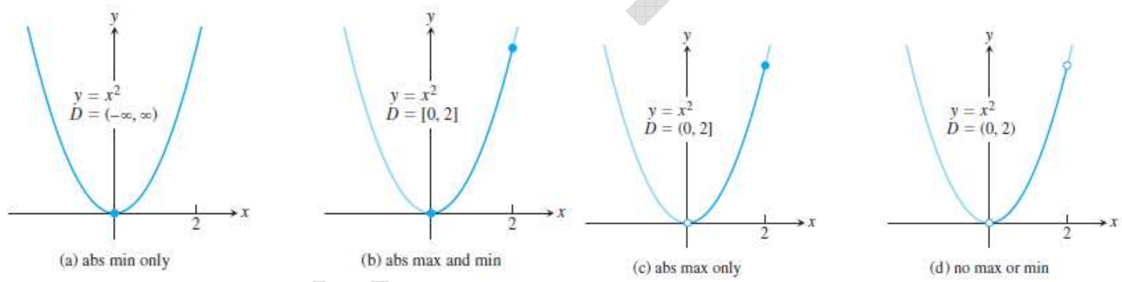
(a) Then f has an ***absolute maximum value*** at c if $f(x) \le f(c)$ for all $x \in D$, and $f(c)$ is the absolute maximum value of f on D.

(b) Then f has an ***absolute minimum value*** at c if $f(x) \ge f(c)$ for all $x \in D$, and $f(c)$ is the absolute minimum value of f on D.

[Sometimes the word ‘global’ is used instead of ‘absolute’.)

Absolute maximum and minimum values are called ***absolute extrema.***

Example:



(T*)

a) $y = x^2, x \in (-\infty, \infty)$: no absolute maximum, absolute minimum of 0 at $x = 0$

b) $y = x^2, x \in [0,2]$: absolute maximum of 4 at $x = 2$, absolute minimum of 0 at $x = 0$

c) $y = x^2, (0,2]$: absolute maximum of 4 at $x = 2$, no absolute minimum

d) $y = x^2, (0,2)$: no absolute extremum.

Definition: Local (or **relative) Extreme Value**

A function f has a ***local maximum value*** at c within its domain D_f if there exists an open interval $I \subseteq D_f$ containing c such that

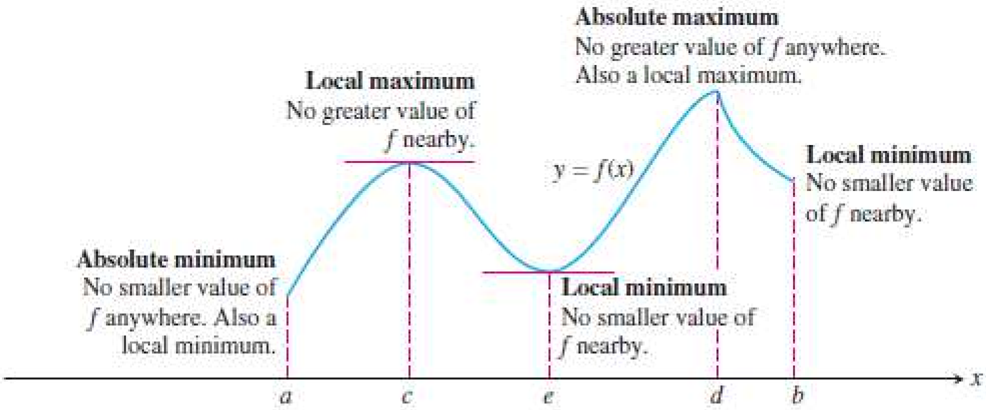
$$f(x) \le f(c) \quad \text{for all } x \in I.$$

A function f has a ***local minimum value*** at c within its domain D_f if there exists an open interval $I \subseteq D_f$ containing c such that

$$f(x) \ge f(c) \quad \text{for all } x \in I.$$

Local maximum and minimum values are called ***local extrema***.

We can extend the definition of local *extrema* to the endpoints of a closed interval by defining *f* to have a local maximum or local minimum value at the endpoint *c* if the appropriate inequality holds for all *x* in some half-open interval with *c* as an endpoint.



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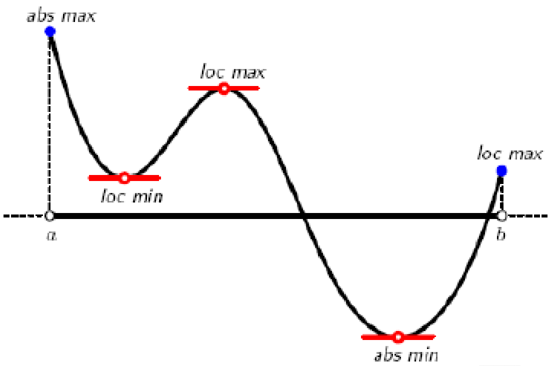
How to identify types of maxima and minima for a function with domain [a, b]



<u>Extreme Value Theorem for Continuous Functions</u>

If *f* is continuous on [*a*,*b*], then there exist numbers *c* and *d* in [*a*,*b*] such that *f* has an absolute maximum value at *c* and an absolute minimum value at *d*.

<u>Fermat's Theorem: Local Extreme Value</u>

If *f*(*x*) has a local extreme value at $x = c$ and *f*(*x*) is differentiable at $x = c$ (i.e., $f'(c)$ exists) then $f'(c) = 0$.

<u>Example:</u>

a) $f(x) = x^3 - 3x$ has local maximum at $x = -1$ and local minimum at $x = 1$. In fact, $f'(x) = 3x^2 - 3$ is equal to zero at $x = \pm 1$.

The graph of $y = x^3 - 3x$

<u>Remark.</u>

The converse to Fermat's Theorem does not hold; that is, if $f'(c) = 0$ it does not necessarily follow that *f* has a local *extremum* at *c*. [***Example***?] ***Can you suggest?***

For a differentiable function, "$f'(c) = 0$" is <u>a necessary condition</u>, but <u>NOT a sufficient condition</u> for the existence of a local extremum at *c*.

<u>Definition: Critical Point</u>

Let $f(x)$ be a function with domain D_f. A point $c \in D_f$ is called a ***critical point*** of f if either $f'(c) = 0$ or $f'(c)$ fails to exist.

[Sometimes, "critical point" is referred to as "critical number".]

<u>Definition: Stationary Point</u>

Let $f(x)$ be a function with domain D_f. A point $c \in D_f$ is called a ***stationary point*** of f if $f'(c) = 0$.

Fermat's Theorem can also be restated as:

If f has a local maximum or minimum at c, then c is a critical point of f.

For a function continuous on a closed interval [a, b] and differentiable on (a, b), every local maximum or minimum will be at one of the endpoints, or at a stationary point of f.

To find an absolute maximum or minimum of a function continuous on a closed interval [a, b] and differentiable on (a, b), we note that either it is local [in which case it occurs at a stationary point] or it occurs at an endpoint of the interval.
Thus the following three-step procedure always works.

<u>To find the absolute maximum and minimum values of a function f continuous on a closed interval [a, b] and differentiable on (a, b),</u>

1. Find the stationary points of f and the values of f at these points.
2. Find the values of f at the endpoints of the interval.
3. From the values of f obtained in steps 1 and 2, determine the absolute extrema.

[Note: Steps 1 & 2 could be interchanged.]

<u>Example:</u>

Find the absolute extrema (i.e. the absolute maximum and minimum values of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ for $x \in [-3, 3]$.

<u>Solution</u>:

The function f is continuous on the closed interval $[-3, 3]$,

and f is differentiable with $f'(x) = 6x^2 - 6x - 12$.

When $f'(x) = 0$, $6x^2 - 6x - 12 = 0$, i.e., $6(x^2 - x - 2) = 6(x-2)(x+1) = 0$

So, $f'(x) = 0$ when $x = -1$ or $x = 2$. [*stationary points*]

Checking values of f at –1, 2 and the endpoints of the interval $[-3, 3]$:

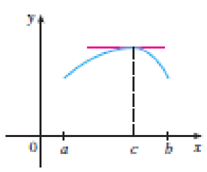
$f(-1) = \ldots = 8$, $f(2) = \ldots = -19$, $f(-3) = \ldots = -44$, $f(3) = \ldots = -8$

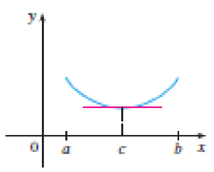
Therefore, the absolute maximum value is $f(-1) = 8$ and the absolute minimum value is $f(-3) = -44$.

Many results used in this chapter are results of the Mean Value Theorem. We state here Rolle's Theorem which is used to prove the Mean Value Theorem.

<u>Rolle's Theorem</u>

Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b)$, then there is at least one number $c \in (a,b)$ such that $f'(c) = 0$.



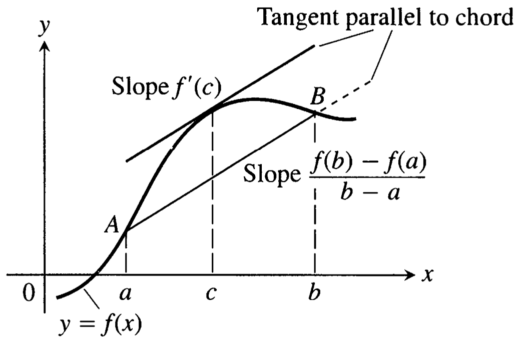


[*The theorem does not help you in finding c*.]

The Mean Value Theorem

Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

[*The theorem does not help you in finding c*.]



[***For this course***, *we would not discuss problems that involve the direct use of the Mean Value Theorem*.]

Increasing, Decreasing and Monotonic Functions

Definitions

A function f is said to be ***increasing on an interval*** I if

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

A function f is said to be ***decreasing on an interval*** I if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

A function that is increasing or decreasing on I is called ***monotonic*** on I.

(Some people use '*monotone*')

The First Derivative Test for Monotonic Functions

Suppose that $f = f(x)$ is differentiable on (a,b), then

1. if $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on (a,b),
2. if $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant on (a,b), and
3. if $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on (a,b).

Example:

Where is the function $f(x) = x^3 - 3x$ increasing? Where is it decreasing?

Solution

$f(x) = x^3 - 3x$; $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

We wish to find where $f'(x) > 0$ and where $f'(x) < 0$.

Interval	$(x-1)$	$(x+1)$	$f'(x) = 3x^2 - 3$	f
$x < -1$	–	–	+	Increasing on $(-\infty, -1)$
$-1 < x < 1$	–	+	–	Decreasing on $(-1, 1)$
$1 < x$	+	+	+	Increasing on $(1, \infty)$

The First Derivative Test for Local Extrema:

The following test applies to a differentiable function *f*(*x*):

(a) At a critical point *c*:

(i) if $f'(x)$ changes from positive to negative at *c*, then *f* has a ***local maximum*** *value* at *c*.

(ii) if $f'(x)$ changes from negative to positive at *c*, then *f* has a ***local minimum*** *value* at *c*.

(iii) if $f'(x)$ has the same sign on both sides of *c*, *then f* has **no** *local extrema value* at *c*.

(b) At a left endpoint *a*:

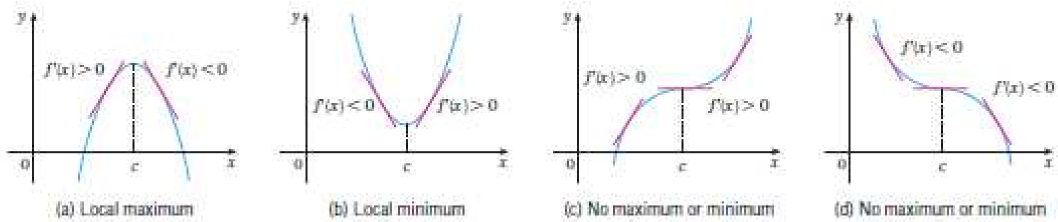
If $f'(x)<0$ for *x* near *a* with $x>a$ then f has a *local maximum* value at *a*.

If $f'(x)>0$ for *x* near *a* with $x>a$ then f has a *local minimum* value at *a*.

(c) At a right endpoint *b*:

If $f'(x)<0$ for *x* near *b* with $x<b$ then f has a *local minimum* value at *b*.

If $f'(x)>0$ for *x* near *b* with $x<b$ then f has a *local maximum* value at *b*.



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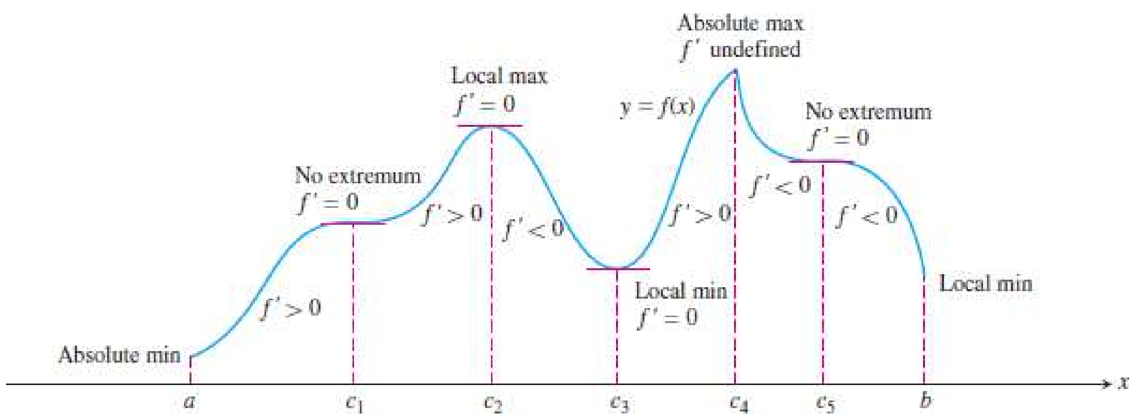

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Example

Find the local maximum and minimum values of the function $f(x)=x^3-3x$ in the interval $[-2.5,2.5]$. Then determine the absolute extrema of *f* on the interval $[-2.5,2.5]$.

Solution

Find the critical/stationary points: $f'(x)=3x^2-3=3(x^2-1)=3(x-1)(x+1)$

$f'(x)=0$ when $x=\pm 1$

The stationary points: $x=\pm 1$

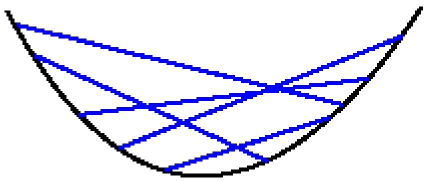
Interval	$(x-1)$	$(x+1)$	$f'(x)=3x^2-3$	f
$-2.5<x<-1$	$-$	$-$	$+$	Increasing on $(-2.5,-1)$
$-1<x<1$	$-$	$+$	$-$	Decreasing on $(-1,1)$
$1<x<2.5$	$+$	$+$	$+$	Increasing on $(1,2.5)$

	At	Value
Local minimum	$-2.5, \quad 1$	$f(-2.5)=\ldots, \quad f(1)=\ldots$
Local maximum	$-1, \quad 2.5$	$f(-1)=\ldots, \quad f(2.5)=\ldots$

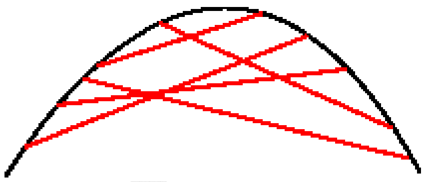
From the local maxima and local minima, we see that for *f* on the interval $[-2.5,2.5]$, the absolute maximum is _______ (occurring at _______) and the absolute minimum is _______ (occurring at _______) .

Definition of Concavity

A function *f* is **convex** (or **concave up**) on an interval if the line segment connecting any two points on the graph of the function lies above the graph between those two points.
A function *f* is **concave** (or **concave down**) on an interval if the line segment connecting any two points on the graph of the function lies above the graph between those two points.



Concave up



Concave down

We can use the second derivative to tell where a function is concave up or concave down.

The Second Derivative Test for Concavity

Let $y=f(x)$ be twice-differentiable on an interval *I*.

(a) If $f''(x)>0$ on I, then the graph of f over *I* is concave up .

(b) If $f''(x)<0$ on I, then the graph of f over *I* is concave down.

Definition of Inflection Point

A point *P* on the graph of a continuous function $y=f(x)$ is called an **inflection point** if the graph changes from concave up to concave down or from concave down to concave up at *P*.

At a point of inflection $(c, f(c))$, $f''(c)=0$ or $f''(c)$ fails to exist.
" $f''(c)=0$ " here is only a necessary condition when the second derivative exists, but not a sufficient condition.

<u>The Second Derivative Test for Local Extrema</u>:

Let $f(x)$ be differentiable on an interval *I* containing $x = a$, with $f'(a) = 0$. Suppose $f'(x)$ is also differentiable near *a* with $f''(x)$ is continuous.

(i) If $f''(a) < 0$, then $x = a$ is a *local maximum*.

(ii) If $f''(a) > 0$, then $x = a$ is a *local minimum*.

(iii) If $f''(a) = 0$, then no conclusion can be drawn regarding extreme values. (i.e., inconclusive)

<u>Example:</u>

1. Find the local maximum and minimum values of the following functions using both the first and second derivative tests.

   a) $y = x^5 - 5x + 3$ b) $y = x + \sqrt{1-x}$

2. Find the extrema of the following functions and the points where the extrema appear..

   a) $y = x^3 - 3x^2 + 3x - 2$ b) $y = \dfrac{x}{x^2+1}$

<u>Solution:</u>

1 a) $y = x^5 - 5x + 3$ [using 2^{nd} derivative test]

At the stationary point, $\dfrac{dy}{dx} = 0$. Therefore

$$5x^4 - 5 = 0$$
$$x = \pm 1$$

To determine whether the stationary point is a maximum or minimum, compute $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = 20x^3$$

At $x = -1$, $\dfrac{d^2y}{dx^2} < 0$, thus the critical point is a local maximum.

When $x = -1$, $y = ... = 7$. So *y* has a local maximum value of 7 at $x = -1$.

At $x = 1$, $\dfrac{d^2y}{dx^2} > 0$, thus the critical point is a local minimum.

When $x = 1$, $y = ... = -1$. So *y* has a local minimum value of -1 at $x = 1$.

(<u>To Students</u>: For comparison, try using the 1^{st} derivative test to find the local extrema for $y = x^5 - 5x + 3$)

1 b)

2 a)

2 b)

7. INDETERMINATE FORMS AND L'HÔPITAL'S RULE

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and that f and g are differentiable on an open interval I containing a. Suppose also that $g'(x) \neq 0$ on I if $x \neq a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \qquad \text{if the limit on the right exists (or is } \infty \text{ or } -\infty\text{).}$$

Remark

When $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, $\lim_{x\to a}\frac{f(x)}{g(x)}$ is said to have the ***indeterminate form*** $\frac{0}{0}$.

L'Hôpital's rule also applies to quotients that lead to the ***indeterminate form*** $\frac{\infty}{\infty}$

If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the latter limit exists. The a here may itself be either *finite* or *infinite*.
[a denotes a real number, ∞ or $-\infty$.]

Example:

Evaluate the following.

a) $\lim_{x\to 1}\frac{3x^2-3}{x^2-x}$ b) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ c) $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}$ d) $\lim_{x\to\infty}\frac{\ln x}{x^{1/3}}$

Solution:

a) $\lim_{x\to 1}\frac{3x^2-3}{x^2-x}$

Let $f(x) = 3x^2-3, g(x) = x^2 - x$

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $g'(x) = 2x-1 \neq 0$ for x near 1.

$$\therefore \lim_{x\to 1}\frac{3x^2-3}{x^2-x} = \lim_{x\to 1}\frac{6x}{2x-1} = \frac{6}{1} = 6$$

b) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2}$$

$$= \lim_{x\to 0}\frac{\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

c) $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}$

$$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x} = \lim_{x\to\infty}\frac{1-4x}{6x+5}$$

$$= \lim_{x\to\infty}\frac{-4}{6} = -\frac{2}{3}$$

(nby, Jun 2017)